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# THE STEADY MOTIONS OF A SYSTEM OF TWO ELASTICALLY COUPLED BODIES†

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The problem of the existence, branching and stability of the steady motions of a system of two elastically coupled bodies in a central gravitational field is considered. Each body is simulated by a weightless rod with point masses at opposite ends. It is assumed that the rods are elastically attached at their mass centres, and the composite body is moving in a plane containing the attracting centre. Both trivial and non-trivial steady motions are studied, on the assumption that none of the principal axes of inertia of the body coincides with the radius vector of the centre of mass or with a tangent to the orbit; it is also assumed that the rods are not orthogonal to one another. The stability of all steady motions is fully investigated and an atlas of bifurcation diagrams presented.


Previous studies have considered the steady motions of two rigidly coupled rods [1] and of two point masses on a spring [2].

1. We consider the linear and rotational motion of a system of two elastically coupled bodies in a central gravitational field. Each body is simulated by a weightless rod $d_s$ of length $2a$, at opposite ends of which are point masses $m_s/2$ $(s = 1, 2)$. It is assumed that the rods are elastically linked together at their centres of mass, and that the composite body is moving in a plane containing the attracting centre.

The position of the composite body is uniquely defined by four generalized coordinates: the distance $r$ from the centre of mass $G$ of the body to the attracting centre $O$, the angles $\varphi_1$ and $\varphi_2$ between the straight line $OG$ and the rods $d_1$ and $d_2$, respectively, and the angle $\varphi$ between a certain fixed direction in the plane of motion and the straight line $OG$.

The kinetic and potential energies ($T$ and $V$, respectively) are given by

$$2T = m(\dot{r}^2 + r^2\dot{\varphi}^2) + m_1a^2(\dot{\varphi}+\dot{\varphi}_1)^2 + m_2a^2(\dot{\varphi}-\dot{\varphi}_2)^2$$

$$2V = -fM\sum_{i=1}^{2} m_i(F_{i+} + F_{i-}) + V_e, \quad F_{i\pm} = (r^2 + a^2 \pm 2ar\gamma_i)^{-1/2}, \quad \gamma_i = \cos\varphi_i$$

where $M$ is the mass of the attracting body, $m = m_1 + m_2$ is the mass of the body, $f$ is the gravitational constant, and $V_c$ is the potential energy of deformation. We will assume that $V_e = k(\varphi_1 + \varphi_2 - \pi/2)^2$, where $k$ is the stiffness of the elastic coupling between the rods.

The Lagrangian $L = T - V$ is independent of the angle $\varphi$. Consequently, the equations of motion, in addition to the energy integral $T + V = h$, admit of the integral

$$\partial T / \partial\dot{\varphi} = \varkappa = \text{const} \tag{1.1}$$

and the body may execute motions of the form

$$r = r^\circ = \text{const}, \quad \varphi_1 = \varphi_1^\circ = \text{const}, \quad \varphi_2 = \varphi_2^\circ = \text{const}, \quad \dot{\varphi} = \omega^\circ = \text{const} \tag{1.2}$$

In such motion, the centre of mass of the body moves at a uniform velocity in circles about the attracting centre; the body maintains a fixed orientation relative to the centre and the angle between the rods is constant.

Ignoring the cyclic variable $\varphi$, introducing the Routh function $R = R(\dot{r}, \dot{\varphi}_1, \dot{\varphi}_2, r, \varphi_1, \varphi_2, \varkappa) = T - V - \varkappa\dot{\varphi}$ with the variable $\dot{\varphi}$ eliminated with the help of the area integral, and assuming, without loss of generality, that the units are chosen so that $fM = 1$, $a = 1$, $m_1 \geqslant m_2$, we can write the effective potential as

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$$W^* = -R_0 = m_1 V - \frac{\varkappa^2}{2(1+\mu)(r^2+1)}, \quad \mu = m_2 / m_1 \in (0;1]$$

The constants $r^\circ$, $\varphi_1^\circ$, $\varphi_2^\circ$ in (1.2) correspond to critical points of the effective potential, i.e. to critical points of the function $W = W^*/m_1$, while the constant $\omega^\circ$ in (1.2) is determined from (1.1): $\omega^\circ = \varkappa/[m(r^2+1)]$.

**2.** Let us determine the critical points of the function $W$. To that end we consider the following system of equations

$$W_{\varphi_i} = \frac{r \sin\varphi_1}{2} \mu^{i-1}[F_{i-}^3 - F_{i+}^3] + \tilde{k}\left(\varphi_1 + \varphi_2 - \frac{\pi}{2}\right) = 0 \tag{2.1}$$

$$W_r = \frac{1}{2}(G_1 + \mu G_2) - \frac{\varkappa^2 r}{(1+\mu)(r^2+1)^2} = 0 \tag{2.2}$$

$$W_{\varphi_i} = \partial W / \partial \varphi_i, \quad W_r = \partial W / \partial r$$

$$G_i = (r+\gamma_i)F_{i+}^3 + (r-\gamma_i)F_{i-}^3, \quad i = 1,2, \quad \tilde{k} = k / m_1$$

Equations (2.1) are satisfied, identically with respect to $r$, by the values

$$1) \ \varphi_1 = 0, \quad \varphi_2 = \pi/2 \pmod{\pi}$$
$$2) \ \varphi_2 = 0, \quad \varphi_1 = \pi/2 \pmod{\pi}$$

Equation (2.2) then becomes

$$\varkappa^2 = H_i(r) \equiv (1+\mu)\left(p_i(r^2+1)^{1/2} + q_i \frac{(r^2+1)^2}{2r}\left[\frac{1}{(r+1)^2} + \frac{\operatorname{sign}(r-1)}{(r-1)^2}\right]\right), \quad i = 1,2$$

$$(p_1 = q_2 = \mu, \quad p_2 = q_1 = 1)$$

Analysing the functions $H_i(r)$ as was done in [1], we conclude that Eq. (2.2) has no solutions when $\varkappa_{22}^{\circ 2} < \varkappa^2 < \varkappa_{1s}^{\circ 2}$, two families of solutions $r = r_{1s}^{\pm}(\varkappa^2)$ when $\varkappa^2 > \varkappa_{1s}^{\circ 2}$ for all values of $\mu$, and if $\mu \in (0, 0.06)$ there are two further families of solutions $r = r_{22}^{\pm}(\varkappa^2)$ when $H_2(0) < \varkappa^2 < \varkappa_{22}^{\circ 2}$; moreover, $r_{is}^{+}(\varkappa^2) > r_{is}^{\circ} > r_{is}^{-}(\varkappa^2)$ $(i, s = 1, 2)$. Here

$$H_s' > 0 \ (H_s' < 0) \quad \text{for} \quad r = r_{is}^{+}(\varkappa^2) \ (r = r_{is}^{-}(\varkappa^2)), \quad H_s'(r_{is}^{\circ}) = 0$$

$$\overset{\circ}{\varkappa}_{1s} = H_s(\overset{\circ}{r}_{1s}), \quad \overset{\circ}{\varkappa}_{22} = \begin{cases} H_2(\overset{\circ}{r}_{22}), & \mu \in (0; 0{,}06) \\ H_2(0), & \mu \in [0{,}06; 0{,}5) \\ 0, & \mu \in [0{,}5; 1] \end{cases}$$

In all the remaining cases Eq. (2.2) has a unique solution.

Note that $\varkappa_{22}^{\circ 2} < \varkappa_{1s}^{\circ 2}$ $(s = 1, 2)$, and also $r_{11}^{\circ} > r_{12}^{\circ} > r_{22}^{\circ}$, $H_1 > H_2$ for $\mu \in (0, 1)$; $r_{11}^{\circ} = r_{12}^{\circ}$, $H_1 = H_2$ for $\mu = 1$.

Obviously, solutions of the form

$$\varphi_1 = 0, \quad \varphi_2 = \pi/2, \quad r = r_{11}^{\pm}(\varkappa^2) \tag{2.3}$$

$$\varphi_2 = 0, \quad \varphi_1 = \pi/2, \quad r = r_{12}^{\pm}(\varkappa^2), \quad r = r_{22}^{\pm}(\varkappa^2) \tag{2.4}$$

correspond to orientations of the body such that one of its principal central axes of inertia is aligned along the radius vector of the centre of mass and the other along a tangent to the orbit; the rods are then orthogonal to one another.

**3.** We will now determine the nature of the critical points (2.3) of the effective potential. To that end we evaluate the matrix coefficients of the second variation of the function $W$ along the solution (2.3)

$$C_{11}^{(1)}=W_{rr}^{\circ}\equiv\frac{r}{m(r^2+1)^2}H_1'(r),\quad C_{12}^{(1)}=W_{r\varphi_1}^{\circ}\equiv 0,\quad C_{13}^{(1)}=W_{r\varphi_2}^{\circ}\equiv 0 \tag{3.1}$$

$$C_{22}^{(1)}=W_{\varphi_1\varphi_1}^{\circ}\equiv\frac{r(3r^2+1)}{(r^2-1)^3}+\tilde{k},\quad C_{23}^{(1)}=W_{\varphi_1\varphi_2}^{\circ}\equiv\tilde{k},\quad C_{33}^{(1)}=W_{\varphi_2\varphi_2}^{\circ}\equiv\tilde{k}-\frac{3\mu r^2}{(r^2+1)^{5/2}}$$

The conditions for the steady motions (2.3) to be stable are the inequalities

$$C_{11}^{(1)}>0,\quad C_{22}^{(1)}>0,\quad G^{(1)}=C_{22}^{(1)}C_{33}^{(1)}-(C_{23}^{(1)})^2>0 \tag{3.2}$$

The sign of $C_{11}^{(1)}$ is the same as that of $dH_1/dr$, that is, $C_{11}^{(1)}\gtrless 0$ for $r=r_{11}^{\pm}(\varkappa^2)$, and $C_{22}^{(1)}>0$.

Analysis of the behaviour of $G^{(1)}$ as a function of the parameters $\tilde{k}$, $\mu$ shows that if $k/\mu<3\sqrt{(2)}/8$, then $G^{(1)}=0$ for $r=r_{11}$, and $G^{(1)}<0$ ($G^{(1)}>0$) for $1<r<r_{11}$ ($r>r_{11}$); otherwise, $G^{(1)}>0$ for any $r\in(1,\infty)$.

We can similarly investigate the nature of the critical points (2.4) of the effective potential. The conditions for the steady motions (2.4) to be stable are given by inequalities similar to (3.2) but with the superscript 1 replaced by 2; in (3.1), this replacement must be accompanied by cyclic permutation of the indices $\varphi_1\leftrightarrow\varphi_2$ and replacement of $(r^2-1)^3$ by $|r^2-1|^3$.

The sign of $C_{11}^{(2)}$ is the same as that of $dH_2/dr$, that is $C_{11}^{(2)}\gtrless 0$ for $r=r_{12}^{\pm}(\varkappa^2)$, $C_{22}^{(2)}>0$.

Analysis of the behaviour of $G^{(2)}$ as a function of the parameters $\tilde{k}$ and $\mu$ shows that for certain values of the parameters $G^{(2)}=0$ for $r=r_{12}$ and $r=r_{22}$, and moreover $G^{(2)}<0$ ($G^{(2)}>0$) for $0<r<r_{22}$, $r>r_{12}$ ($r_{22}<r<1$; $1<r<r_{12}$); otherwise, $G^{(2)}>0$ for all $r\in(0,\infty)$.

Our conclusions are thus as follows:

1. If $\tilde{k}<3\sqrt{(2)}/8$, then, depending on the values of $\mu$, we have: (a) two bifurcation points at $r>1$, (b) one double bifurcation point (the two points coincide), (c) no bifurcations.

2. If $\tilde{k}\geqslant 3\sqrt{(2)}/8$, then, depending on the values of $\mu$, we have: (a) one bifurcation point at $r>1$ and one at $0<r<1$; (b) one bifurcation point at $r>1$.

**4.** At the points $\varphi_1=0$, $\varphi_2=\pi/2$, $r=r_{11}$, $\varkappa_{11}^{*2}=H_1(r_{11})$ ($\varphi_1=\pi/2$, $\varphi_2=0$, $r=r_{12}$, $r=r_{22}$, $\varkappa_{12}^{*2}=H_2(r_{12})$, $\varkappa_{22}^{*2}=H_2(r_{22})$) one of the matrix coefficients of the second variation vanishes and the degree of instability of the solutions (2.3) ((2.4)) vanishes. This means that at these points a solution of system (2.1), (2.2) branches off one of these solutions; in this solution the orientation of the body is such that neither of its central axes of inertia coincides with the radius vector of the centre of mass or the tangent to the orbit, and the rods are not mutually orthogonal. We will seek solutions branching off (2.3) in the form

$$1)\ \ \varphi_1=\alpha,\quad \varphi_2=\pi/2-\beta \tag{4.1}$$

and solutions branching off (2.4) in the form

$$2)\ \ \varphi_2=\alpha,\quad \varphi_1=\pi/2-\beta \tag{4.2}$$

For these solutions, system (2.1) is equivalent to the following system of equations

$$\mu=F_i(r,\alpha,\tilde{k}),\quad \mu=R_i(r,\beta,\tilde{k},\mu),\quad i=1,2 \tag{4.3}$$

where

$$F_i(r,\alpha,\tilde{k})=\frac{Q_i(\alpha)}{P_i(f_i(\alpha))}\frac{\zeta(r,P_i(\alpha))}{\zeta(r,Q_i(f_i(\alpha)))},\quad f_i(\alpha)\equiv\beta=\alpha+\delta_i\frac{rQ_i(\alpha)}{2\tilde{k}}\zeta(r,P_i(\alpha))$$

$$R_i(r,\beta,\tilde{k},\mu)=\frac{Q_i(g_i(\beta))}{P_i(\beta)}\frac{\zeta(r,P_i(g_i(\beta)))}{\zeta(r,Q_i(\beta))},\quad g_i(\beta)\equiv\alpha=\beta+\delta_i\frac{\mu rP_i(\beta)}{2\tilde{k}}\zeta(r,Q_i(\beta))$$

$$\zeta(r,x)=(r^2+1-2rx)^{-3/2}-(r^2+1+2rx)^{-3/2}$$

$$Q_1(a)=P_2(a)=\sin a,\quad Q_2(a)=P_1(a)=\cos a,\quad \delta_i=(-1)^{i+1}$$

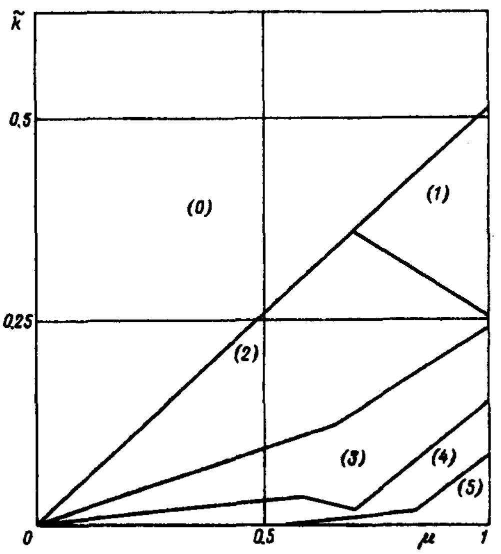


Fig. 1.

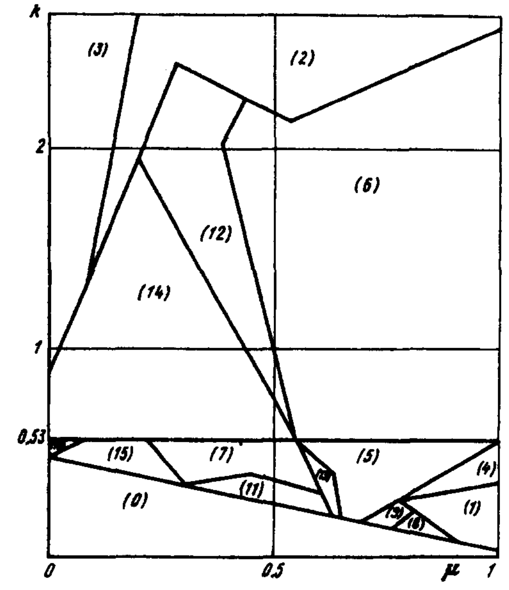


Fig. 2.

When that is the case

$$\varkappa^2 = \frac{(1+\mu)(r^2+1)^2}{2r}(\overline{G}_1 + \mu\overline{G}_2), \quad \overline{G}_i = (r + P_i(\alpha))F_i^3(a) + (r - Q_i(\alpha))F_i^3(-a)$$

Analysis of system (4.3) shows that, depending on the parameters of the problem, one obtains bifurcation diagrams of different possible types: six types for solutions branching off (2.3) and 16 for solutions branching off (2.4).

Figures 1 and 2 demonstrate the domains in the $(\tilde{k}, \mu)$ plane corresponding to the different types of bifurcation diagram. Domains labelled (0) in either figure are those in which there is no bifurcation and those labelled (2) and (3) in Fig. 2 indicate domains with the same type of diagram as in the case of a rigid body [1]. In all other cases the diagrams are essentially distinct from those of a rigid body [1]. In particular, if the parameters of the problem lie in domains (2) and (3) (Fig. 1), the bifurcation diagrams are of the form shown in Figs 3 and 4; but if the parameters lie in domains (6), (12) or (14) (Fig. 2), the bifurcation diagrams are of the form shown in Figs 5, 6 or 7, respectively.

All the bifurcation diagrams are sections of $(r, \varphi_1, \varphi_2, \kappa^2)$ space by hyperplanes (2.3) (Fig. 1) or (2.4) (Fig. 2). The solid curves represent branches lying in these hyperplanes and corresponding to trivial steady motions. The dashed curves indicate projections of branches that lie outside the hyperplanes, corresponding to non-trivial steady motions. Indices 0, 1 and 2 denote the degree of instability of the steady motions of the body corresponding to various orientations. The degree of instability of the non-trivial orientations is indicated in accordance with the general concepts of bifurcation theory.

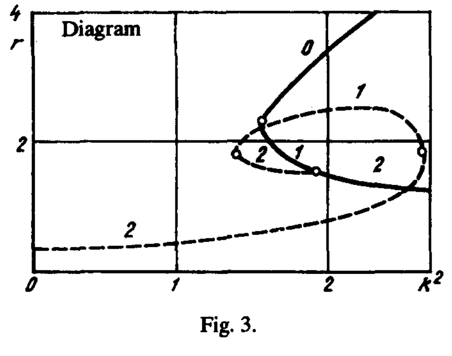


Fig. 3.

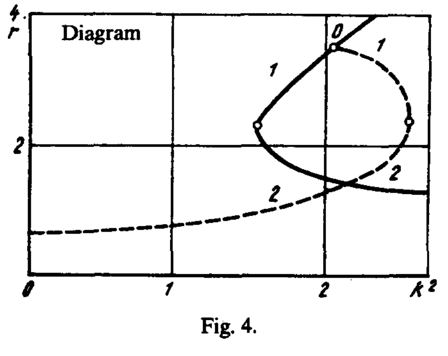


Fig. 4.

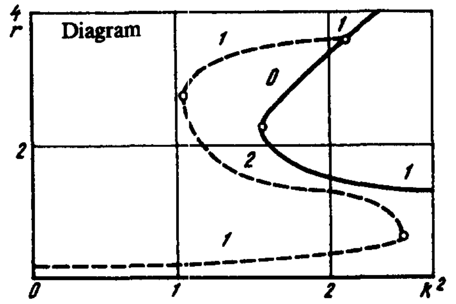


Fig. 5.

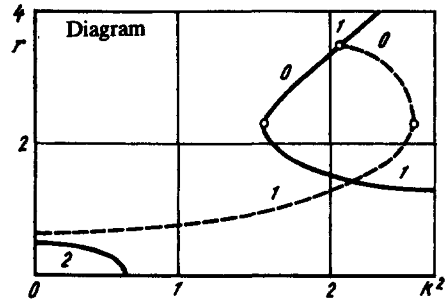


Fig. 6.

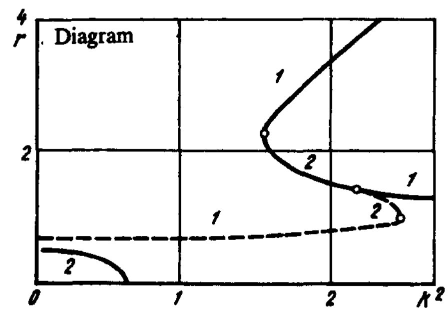


Fig. 7.

Note that if $\tilde{k} \gg 1$, there are no bifurcations for solutions (2.3) (Fig. 1), while only two distinct types of bifurcation diagram are possible for (2.4) (this result is analogous to the previous results of [1]). At the same time, as $\tilde{k}$ is decreased, there are some values of $\mu$ at which (unlike the previous results in [1]) there may be non-trivial steady motions branching off from solutions (2.4) that are stable in the secular sense.

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## REFERENCES

1. ABRAROVA, Ye. V. and KARAPETYAN, A. V., The steady motions of a rigid body in a central gravitational field. *Prikl. Mat. Mekh.*, 1994, **58**, 5, 68–73.
2. BUROV, A. A. and STEPANOV, S. Ya., On mass geometry in the dynamics of deformable bodies. In *Problems in the Investigation of the Stability and Stabilization of Motion*. Vychisl. Tsentr Ross. Akad. Nauk, Moscow, 1995, 107–130.

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